SOME EXTREMAL PROBLEMS OF HEAT TRANSFER IN A LAYER

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Three extremal problems of heat transfer in a layer are formulated for the condition of reverse motion of the thermal medium. The results of numerical solutions of the problems are presented in graphical form.

Wide use is made in industry of the blowing method for thermal processing of porous materials. To obtain a more uniform temperature distribution along the height of the material being treated and to accelerate the thermal process to which it is subjected, in continuous-action assemblies it is frequently the practice to change the direction of blowing of the thermal medium in adjacent zones. This procedure is adopted, for example, in the drying and heat treating of a mineralized coating [1]. It is necessary to have a basis for choosing the ratio of zone lengths in the chambers for heat treating mineralized plates.

We consider a two-zone chamber for the heat treatment of a porous material, as indicated in Fig. 1. The speed of movement of the material being processed is constant and equal to w; then the blowing time of the layer in the first zone is $\tau_1 = l_1/w$, whilst in the second it is $\tau_2 = \tau_0 - \tau_1 = (l_0 - l_1)/w$, where τ_0 is the total heat-treatment time. In designing such an assembly the following problems arise.

1. For what value of τ_1 (τ_0 is fixed) will the maximum heat be assimilated by the material; or, equivalently, when will the maximum mean temperature of the material be attained ?
2. For what value of τ_1 (τ_0 is fixed) will the minimum temperature along the height of the layer be maximum at the instant that the heating process terminates ?

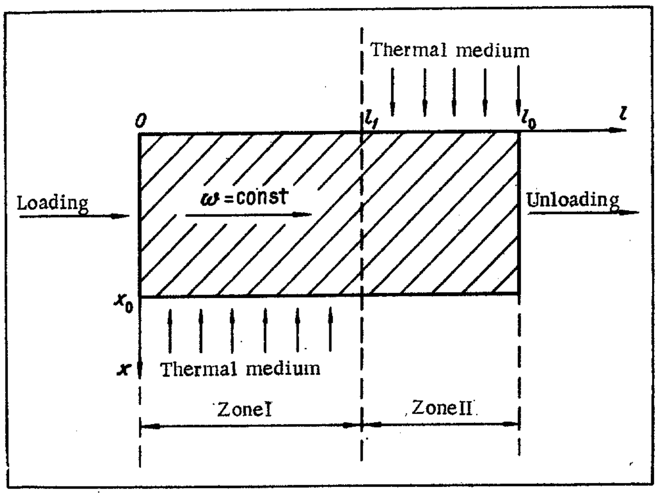

Fig. 1. Schematic of a two-zone chamber for heating a porous material by blowing through of a thermal medium.

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3. What choice of the ratio of τ_1 and τ_0 should be made to ensure that, at the instant of unloading, the the temperature of each point of the layer will be no lower than the technologically assigned temperature for a minimum total time of treatment, $\tau_0 = \min$?

We now formulate these problems exactly, noting that the heating process may be described by known equations for heat transfer in a layer; written in dimensionless form, they are [2, 3]:

$$\frac{\partial\Theta}{\partial Z} = \Theta_g - \Theta; \quad 0 \leqslant Z \leqslant Z_0; \quad \Theta = \Theta(F, Z),$$
$$-\frac{\partial\Theta_g}{\partial F} = \Theta_g - \Theta; \quad 0 \leqslant F \leqslant F_0; \quad \Theta_g = \Theta_g(F, Z) \tag{1}$$

with the initial and boundary conditions

$$\Theta(F, 0) = 1;$$
$$\Theta_g(F_0, Z) = 0 \quad \text{for } 0 \leqslant Z \leqslant Z_1;$$
$$\Theta_g(0, Z) = 0 \quad \text{for } Z_1 \leqslant Z \leqslant Z_0. \tag{2}$$

The transition to dimensional quantities is accomplished by means of the equations

$$\Theta = \frac{t_g^0 - t(F,Z)}{t_g^0 - t^0}; \quad \Theta_g = \frac{t_g^0 - t_g(F,Z)}{t_g^0 - t^0}, \tag{3}$$

where $F = x\alpha_v/mw_g c_g$ is a thickness number; $Z = \alpha_v(\tau - x/w_g)/(1-m)c$ is a time number. We observe that to the maximum layer temperature t there corresponds the minimum dimensionless temperature Θ. We neglect the quantity x/w_g in comparison with τ, assuming that $\tau \gg x/w_g$. This assumption holds for the majority of practical problems. For example, for a mineralized coating, $\tau \approx 200\ x/w_g$. We thus compute the time number Z from the equation

$$Z = \frac{\alpha_v}{(1-m)c}\tau. \tag{4}$$

Moreover in our calculations we obtain somewhat excessive values of the total heating time τ_0. The dimensionless temperature distribution in the layer at the instant of change in the blow-through direction at $Z = Z_1$ has the form [3]

$$\Theta(F, Z_1) = \exp[-(F_0 - F + Z_1)]\, I_0\left(2\sqrt{(F_0 - F)Z_1}\right) + \int_0^{F_0 - F} \exp[-Z_1 - \eta]\, I_0\left(2\sqrt{Z_1\eta}\right) d\eta. \tag{5}$$

Taking this as the initial temperature distribution, we determine, using a result given in [4], the temperature of the layer at the instant $Z = Z_0$ that the process terminates:

$$\Theta(F, F_0, Z_1, Z_0) = \exp(-Z_0)\left\{\Pi(Z_1, F_0 - F) + \int_0^F \Psi(Z_0 - Z_1, \eta)\, \Pi(Z_1, F_0 - F + \eta)\, d\eta\right\}. \tag{6}$$

Here and henceforth,

$$\Pi(Z, F) = \exp(-F)\, I_0\left(2\sqrt{FZ}\right) + \int_0^F \exp(-\xi)\, I_0\left(2\sqrt{Z\xi}\right) d\xi;$$
$$\Psi(Z, F) = \exp(-F)\, I_1\left(2\sqrt{ZF}\right)\sqrt{Z/F}; \tag{7}$$

I_0 and I_1 are Bessel functions of an imaginary argument.

In accord with the problems stated above, our interest centers on the following extremal properties of the temperature distribution (6):

1. For what value of $Z_1 \in [0, Z_0]$ (F_0, Z_0 are given) does the function

$$\Phi_0(Z_1) = \int_0^{F_0} \Theta(F, F_0, Z_1, Z_0)\, dF \tag{8}$$

assume its minimum value?

2. For what value of Z_1(F_0, Z_0 are given) does the function

$$\Phi_1(Z_1) = \max_{F \in [0, F_0]} \Theta(F, F_0, Z_1, Z_0). \tag{9}$$

attain its minimum?

3. What must be the values of Z_1 and Z_0 (F_0 fixed) in order that the function (9) attain a given value $\Phi_1 = \Phi_{1\,given}$ for the smallest possible value of Z_0?

It is not hard to see that the solution of Problem 3 can be obtained if the solution of Problem 2 is known. The procedure is to change the given value Z_0 in Problem 2 as long as the function (9) has not assumed a given value.

1. First of all, we note that for a one-sided blow-through into the layer a strictly monotonic temperature distribution is established, whereby the layers positioned close to the plane of blowing of the thermal medium are heated to a very high temperature.

That for arbitrary Z_1, the function $\Theta(F, Z_1)$, defined by Eq. (5), is a strictly monotonically increasing function of its argument F may easily be confirmed by a direct verification of the sign of its derivative.

Thus the main question in connection with Problem 1 may be reduced to the following. If as a result of a one-sided blow-through into the layer a strictly monotonic temperature distribution is established, from what side should the blow-through be produced to obtain the maximum mean temperature in the material of the layer?

We consider the simplest model. Assume that the initial temperature of the layer changes linearly through its thickness:

$$t(0,\ x) = t^0 + t' + k(x-1);\quad 0 \leqslant x \leqslant 2 \tag{10}$$

and that, after a calculated interval of time, its distribution remains the same. The temperature in the middle of the layer is $t(1) = t'$. Gas blowing into the layer takes place at $x = 0$. It is obvious that for $k < 0$ we have blowing from the "hot" side of the layer, and for $k > 0$ from the "cold" side. With a unit heat-transfer coefficient we have the following equation for the temperature of the gases:

$$-\frac{dt_g}{dx} = t_g - t^0 = t_g - t' - k(x-1);\qquad t_g(0) = t_g^0;\quad 0 \leqslant x \leqslant 2, \tag{11}$$

the solution of which is obviously of the form

$$t_g(x) = (t_g^0 - t' + 2k)\exp(-x) + (t' + kx - 2k). \tag{12}$$

In addition, the amount of heat given off by the gas is determined by the expression

$$Q = \int_0^2 (t_g - t^0)\,dx = (t_g^0 - t')\,[1 - \exp(-2)] - 2k\exp(-2). \tag{13}$$

From this it is evident that the gas releases the maximum amount of heat when $k < 0$, i.e., for blow-through from the "hot" side.

This holds true even in the general case of an arbitrary strictly monotonic initial temperature distribution in the layer.

Since each change in the blow-through direction, at least in the initial stage of the process, leads to blowing of the thermal medium from the "cold" side of a strictly monotonic temperature distribution in the layer, we may expect a lowering of the mean temperature of the layer in comparison with the case in which the blow-through is continued in the same direction. The numerical calculations of the values of the function (8), which we carried out for various values of Z_1, confirm this result.

The values of the function $\Phi_0(Z_1)$ obtained in these calculations are presented in Table 1. We recall that to the minimum of the function $\Phi_0(Z_1)$, $Z_1 \in [0,\ Z_0]$, which is attained on the boundaries of the domain where the function is defined $\{Z_1 = 0;\ Z_1 = Z_0\}$, there corresponds the maximum of the dimensional mean temperature of the material in the layer.

TABLE 1. Values of the Function $\Phi_0(Z_1)$

Z_1	0,0	0,5	1,0	1,5	2,0	2,5	3,0
$F_0=1;\ Z_0=3$	0,133990	0,142661	0,146338	0,147831	0,146337	0,142661	0,133990
$F_0=3;\ Z_0=3$	0,318705	0,351521	0,372277	0,378775	0,372278	0,351521	0,318705

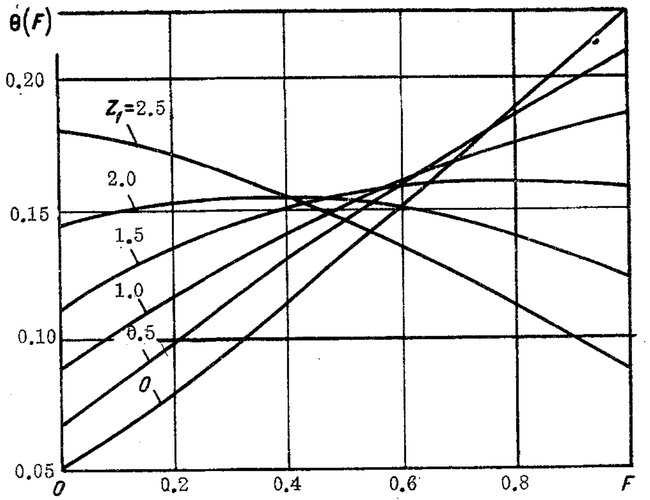

Fig. 2. Dimensionless temperature distribution over the layer at various lengths of the first treatment zone Z_1; $F_0 = 1$, $Z_0 = 3$.

It is of interest that the function is symmetric with respect to the point $Z_1 = Z_0/2$.

We have not been successful in obtaining a rigorous proof, by means of an analytical study, that the function (8) is symmetric. We note that for points which are symmetric about the value $Z_1 = Z_0/2$ only the values of the function $\Phi_0(Z_1)$ are equal, i.e., the areas under the temperature distributions obtained, which are themselves temperature distributions, differ substantially from one another. This is shown in Fig. 2, where distributions of the dimensionless temperature of the layer are presented for Example 1. The distributions for the symmetric values $Z_1 = 0.5$, $Z_1 = 2.5$, and $Z_1 = 1.0$; $Z_1 = 2.0$, which correspond to one and the same mean temperature, are: $\Phi_0(0.5) = \Phi_0(2.5) = 0.142661$, $\Phi_0(1.0) = \Phi_0(2.0) = 0.146337$; from the point of view of the temperature level attained (the value of the function $\Phi_1(Z_1)$) the values are substantially different: $\Phi_0(0.5) = 0.20795$; $\Phi_1(2.5) = 0.18168$; $\Phi_1(1.0) = 0.18464$; $\Phi_1(2.0) = 0.15452$.

2. In formulating an algorith for calculating the values of $\Phi_1(Z_1)$ for an arbitrary fixed value of Z_1, study of the derivative of the function $\Theta\,(F,\ F_0,\ Z_1,\ Z_0)$ with respect to F plays an essential role; this derivative is given by

$$f(F) = \frac{d\Theta(F,\ F_0,\ Z_1,\ Z_0)}{dF} = \exp(-Z_0)\Big\{\Pi(Z_1,\ F_0)\,\Psi(Z_0 - Z_1,\ F)$$

$$- \Psi(Z_1,\ F_0 - F) - \int_0^F \Psi(Z_0 - Z_1,\ \eta)\,\Psi(Z_1,\ F_0 - F + \eta)\,d\eta\Big\}. \tag{14}$$

It may be verified directly that the function f(F) has not more than one zero on the interval $[0,\ F_0]$ for an arbitrary fixed value of Z_1. Moreover, if there exists a point F^*, satisfying the condition $f(F^*) = 0$, then at this point the function $\Theta(F)$ attains a strict maximum with respect to F.

Using this result, we can readily formulate an algorithm for calculating the function $\Phi_1(Z_1)$ for arbitrary Z_1.

A search for the minimum of the function $\Phi_1(Z_1)$ with $Z_1 \in [0,\ Z_0]$ may be organized by means of an arbitrary known method of search for the extremum of a unimodal function. In our calculations we used a search method over discrete points (Fibonacci's method) [5].

The value of Z_1^0 (the minimum point of the function (9)) was calculated with an accuracy of 10^{-3}. The integrals in Eqs. (6) and (13) were computed by Gauss' method for n = 14 [6]. The calculations were performed on the M-220 computer in the range $1 \leq F_0 \leq 10$; $0.01 \leq \Theta \leq 1.00$.

Results of the calculations appear in Figs. 3 and 4.

For given values of the numbers F_0 and Z_0, according to Fig. 3, a limiting temperature level Θ_m $\Phi_1(Z_1^0)$ may be determined, which is attainable in a two-zone chamber; according to Fig. 4, we can determine the ratio Z_1^0/Z_0, which is the ratio of the length of the first zone to the total chamber length in an optimum construction, and also the coordinate F_m of the point with minimum temperature.

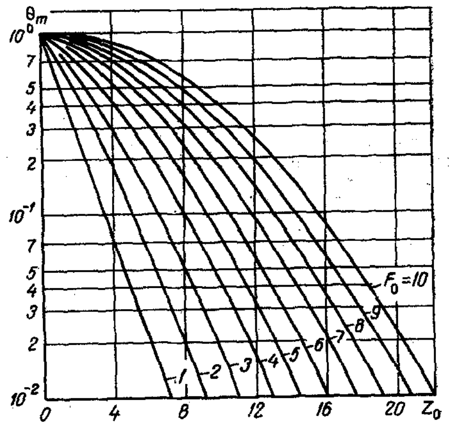

Fig.3. Graphs for calculation of minimum temperature Θ_m in layer section when material is treated in a two-zone chamber with an optimum zone length ratio $\Theta_m = (t_g^0 - t(F_m, F_0, Z_1^0, Z_\rho))/(t_g^0 - t^0)$.

3. The algorithm for solving this problem with the aid of Figs. 3 and 4 is as follows. For given values of Θ_m and F_0 we determine (Fig. 3) the total layer heating time Z_0 (total chamber length) and then, knowing the values of F_0 and Z_0, we determine the ratio Z_1^0/Z_0 and the coordinate F_m (Fig. 4) of the cold point of the layer itself.

We consider first the calculation of a two-zone chamber for heat treatment of a mineralized coating of minimum length (Problem 3). The initial data are: $x_0 = 0.1$ m; $w_g = 0.4$ m/sec; $\alpha_0 = 20 \cdot 10^3 \times 1.163$ kJ /m^3 · sec · deg; $c_g = 4190 \times 0.3$ kJ/m^3 · deg; $c = 600 \times 4190$ kJ/m^3 · deg; $m = 0.96$; $t^0 = 20$°C; $t_g^0 = 210$°C.

We wish to calculate the zone length ratio of a two-zone heat-treatment chamber which ensures heating of each point of the coating to a given temperature $t_m = 180$°C in the minimum time. From Figs. (3) and (4) we determine the values of the thickness number $F_0 = 4.85$ and the temperature $\Theta_m = 0.158$; from Fig. 3 we find $Z_0 = 7.80$. Next, from Fig. 4 we have $F_m = 0.24$; $Z_1^0 = 0.86$.

For a single-zone heat-treatment chamber the given temperature in this example is reached only for $Z_0 = 9.2$. When the zone lengths are equal, i.e., when $Z_1 = Z_0/2$, the value $\Theta_m = 0.158$ is reached only for $Z_0 = 9.6$, i.e., in both cases we have an increase in the total heating time of roughly 20% in comparison with the optimum.

We consider yet another example. Assume we have the following fixed values: $F_0 = 7.0$ and $Z_0 = 9.0$. What will be the maximum dimensionless temperature Θ_m at a layer section for various zone length ratios?

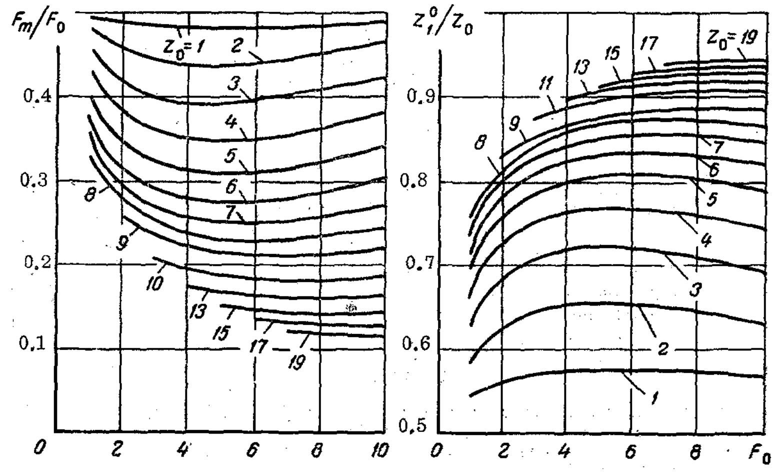

Fig. 4. Graphs for calculation of residence time of material in first zone (Z_1^0) and for calculation of coordinate F_m of point with minimum temperature for a two-zone chamber with an optimum zone length ratio.

I. One-sided blow-through, $Z_1 = 0$; $\Theta_m = 0.354$.

II. Two-zone chamber for $Z_1 = Z_0/2 = 4.5$; $\Theta_m = 0.331$.

III. Optimum zone ratio, $Z_1 = Z_1^0 = 0.88Z_0 = 7.9$; $\Theta_m = 0.245$.

A comparison of the results obtained in the two-zone heat treatment of a layer with an optimum zone length ratio with the one-zone and two-zone treatment with equal zones shows that the saving in heating time of a layer to a given temperature, resulting from optimization, amounts to 5 to 25%, depending on the initial criteria of the problem.

For a significant layer "thickness" ($F_0 > 5$) and for large heating "times" $Z_0 > 20$, the optimum zone ratio, as can be seen from Fig. 4, is $Z_1^0/Z_0 \approx 0.95$. In this region the two-zone chamber with equal zones results in significant losses at a definite layer temperature. Thus, for $F_0 = 9$ and $Z_0 = 21$, in the single-zone chamber we have a maximum dimensionless temperature $\Theta_m = 0.01606$; in the two-zone chamber, with $Z_1 = Z_0/2 = 4.5$, we have $\Theta_m = 0.0843$; for the optimum zone ratio, $Z_1 = Z_1^0 = 0.94Z_0$; $\Theta_m = 0.00981$.

The graphs presented here can be used both for designing two-zone heat-treatment chambers for heat treating porous materials by a blow-through process and also for estimating the effectiveness of operating assemblies of a similar type.

NOTATION

τ	is the time;
x	is the coordinate along the height of the layer;
l_1, l_0	are the length of the first zone of the thermal-treatment chamber and the overall length of the chamber;
τ_1, τ_0	are the time of treatment in the first zone and the overall heating time of the layer;
$t(x, \tau)$	is the temperature of the layer and the heat-transfer agent;
t_g^0, t^0	are the temperature of the heat-transfer agent at the entrance to the layer and the initial temperature of the layer;
t_m	is the minimum temperature in the layer section;
x_m	is the coordinate of the point with the minimum temperature;
τ_1^0	is the optimum time of treatment in the first zone;
x_0	is the thickness of the layer;
c_g, c	are the specific heat capacity of the heat-transfer agent and the material of the layer;
m	is the coefficient of porosity;
α_v	is the volume heat-transfer coefficient;
w_g	is the rate of injection of the heat-transfer agent;
w	is the rate of passage of the material through the treatment zones;
k	is the angle of inclination of the linear temperature distribution in the layer;
t'	is the mean temperature of the layer.

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